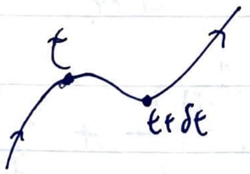
Change of coordinates (Lamb)

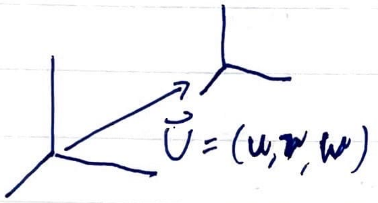
$$F(x+u\delta t,\ y+v\delta t,\ z+w\delta t,\ t+\delta t)$$
$$= F + u\delta t\frac{\partial F}{\partial x} + v\delta t\frac{\partial F}{\partial y} + w\delta t\frac{\partial F}{\partial z} + \delta t\frac{\partial F}{\partial t} \quad (1)$$



material derivative: $\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}$

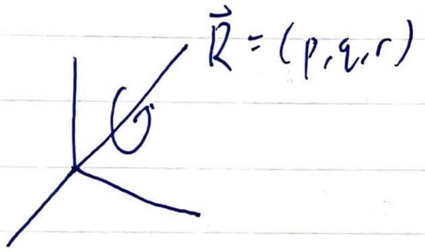
moving axis

$\frac{D\vec{x}}{Dt} = \vec{u} - \vec{U}$



$$\frac{Dx}{Dt} = u - U \qquad \frac{Dy}{Dt} = v - V \qquad \frac{Dz}{Dt} = w - W$$

$\frac{D\vec{x}}{Dt} = \vec{u} + \vec{x}\times\vec{R}$



$$\frac{Dx}{Dt} = u + ry - qz \qquad \frac{Dy}{Dt} = v + pz - rx$$
$$\frac{Dz}{Dt} = w + qx - py$$

$\frac{D\vec{u}}{Dt} = \frac{\partial\vec{u}}{\partial t} - \vec{u}\times\vec{R}$

$$\frac{\partial u}{\partial t} - rv + qw + \frac{\partial u}{\partial x}\frac{Dx}{Dt} + \frac{\partial u}{\partial y}\frac{Dy}{Dt} + \frac{\partial u}{\partial z}\frac{Dz}{Dt}$$
$$\frac{\partial v}{\partial t} - pw + ru + \frac{\partial v}{\partial x}\frac{Dx}{Dt} + \frac{\partial v}{\partial y}\frac{Dy}{Dt} + \frac{\partial v}{\partial z}\frac{Dz}{Dt}$$
$$\frac{\partial w}{\partial t} - qu + pv + \frac{\partial w}{\partial x}\frac{Dx}{Dt} + \frac{\partial w}{\partial y}\frac{Dy}{Dt} + \frac{\partial w}{\partial z}\frac{Dz}{Dt}$$

} acceleration (2)

Polar Co-ordinates (r, θ)

$$\frac{Du}{Dt} \rightarrow \frac{Du}{Dt} - \frac{v}{r}v \qquad \frac{Dv}{Dt} \rightarrow \frac{Dv}{Dt} + \frac{v}{r}u$$

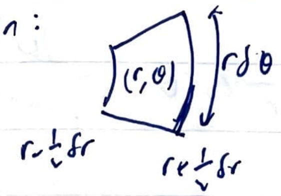
Change of coordinates (Lamb)

let $x \to r$ $y \to \theta$

$u \to u$ $v \to \omega = \frac{v}{r}$

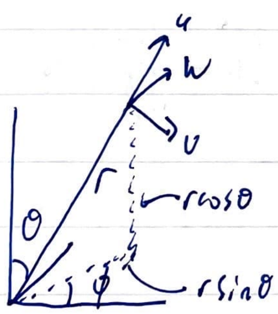
$$\therefore \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + v\frac{\partial}{r\partial\theta} \quad \text{from (1)}$$

expansion:



$$\left(u+\frac{\partial u}{\partial r}\frac{1}{2}\delta r\right)\left(r+\frac{1}{2}\delta r\right)\delta\theta$$
$$-\left(u-\frac{\partial u}{\partial r}\frac{1}{2}\delta r\right)\left(r-\frac{1}{2}\delta r\right)\delta\theta$$
$$+\left(v+\frac{\partial v}{\partial\theta}\frac{1}{2}\delta\theta\right)\delta r-\left(v-\frac{\partial v}{\partial\theta}\frac{1}{2}\delta\theta\right)\delta r$$
$$\approx\left(u+\frac{\partial u}{\partial r}r+\frac{\partial v}{\partial\theta}\right)\delta r\,\delta\theta$$
$$\therefore \Delta = \frac{u}{r}+\frac{\partial u}{\partial r}+\frac{\partial v}{r\partial\theta}$$

Spherical Co-ordinates (r,θ,ϕ)



$v\,\delta t = r\,\delta\theta$

$w\,\delta t = r\sin\theta\,\delta\phi$

$u\,\delta t = \delta r$

instantaneous movement

u-axis : $\cos\theta\,\delta\phi$

v-axis : $-\sin\theta\,\delta\phi$

w-axis : $\delta\theta$

$\vec{\omega} = (p,q,r)$

$$\therefore p = \frac{w}{r}\cot\theta$$
$$q = -\frac{w}{r}$$
$$r = \frac{v}{r}$$

From (2),

$$\frac{Du}{Dt} - rv + qw = \frac{Du}{Dt} - \frac{v^2+w^2}{r}$$
$$\frac{Dv}{Dt} - pw + ru = \frac{Dv}{Dt} + \frac{uv}{r} - \frac{w^2}{r}\cot\theta$$
$$\frac{Dw}{Dt} - qu + pv = \frac{Dw}{Dt} + \frac{wu}{r} + \frac{vw}{r}\cot\theta$$

Change of coordinates (Lamb)

let $x \to r$ $\quad y \to \theta$ $\quad z \to \phi$

$u \to u$ $\quad v \to \omega_\theta = \frac{v}{r}$ $\quad w \to \omega_\phi = \frac{w}{r\sin\theta}$

$$\therefore \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + v\frac{\partial}{r\partial\theta} + w\frac{\partial}{r\sin\theta\,\partial\phi} \quad \text{from (1)}$$

$$\Delta = \frac{\partial u}{\partial r} + \frac{2u}{r} + \frac{\partial v}{r\partial\theta} + \frac{v}{r}\cot\theta + \frac{\partial w}{r\sin\theta\,\partial\phi}$$

(sketch) proof:
$$(u + \frac{\partial u}{\partial r}\frac{1}{2}\delta r)(r + \frac{1}{2}\delta r)^2 \delta\theta\delta\phi$$
$$-(u - \frac{\partial u}{\partial r}\frac{1}{2}\delta r)(r - \frac{1}{2}\delta r)^2 \delta\theta\delta\phi$$
$$+(v + \frac{\partial v}{\partial\theta}\frac{1}{2}\delta\theta) r\left[\sin\theta + \cos\theta\cdot\frac{1}{2}\delta\theta\right]\delta\phi\,\delta r$$
$$-(v - \frac{\partial v}{\partial\theta}\frac{1}{2}\delta\theta) r\left[\sin\theta - \cos\theta\cdot\frac{1}{2}\delta\theta\right]\delta\phi\,\delta r$$
$$+(w + \frac{\partial w}{\partial\phi}\frac{1}{2}\delta\phi) r\,\delta r\,\delta\theta - (w - \frac{\partial w}{\partial\phi}\frac{1}{2}\delta\phi) r\,\delta r\,\delta\theta$$